JOURNAL OF APPROXIMATION THEORY 6, 366–377 (1972)

Bicubic Spline Interpolation in Rectangular Polygons

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Communicated by Oved Shisha

DEDICATED TO PROFESSOR J. L. WALSH
ON THE OCCASION OF HIS 75TH BIRTHDAY

1. INTRODUCTION

The extension of the theory of cubic splines to more than one dimension was initiated by Birkhoff and Garabedian [4], and elaborated upon by DeBoor [9] and Ahlberg, Nilson, and Walsh [2]. DeBoor's bicubic spline interpolation scheme [9; p. 214] has become the standard scheme for rectangular regions. However, for more general rectangular polygons, the question of how "best" to characterize bicubic splines appears to be an open question. In [7, 8] and this paper the authors consider this question.

In [7] the space $S_1^2(\mathscr{R}, \pi)$ of bicubic splines over a partitioned rectangular polygon $(\mathscr{R}, \pi)$ was defined (in the most natural way) to be the subspace of $C^{(2)}[\mathscr{R}]$ consisting of functions which reduce to bicubic $(\sum_{i,j=0}^{3} \alpha_{ij}x^iy^j)$ polynomials in each rectangular element $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ of the mesh π. An *algebraically well set* interpolation scheme in the sense of [3; p. 169] was given for $S_1^2(\mathscr{R}, \pi)$ in [7, Theorem 3].

The main purpose of this paper is to consider a bicubic spline interpolation scheme (Theorem 4) which (*unlike* the scheme given in [7; Theorem 3]), is *analytically well set* for a class of uniformly partitioned rectangular polygons. That is, as the mesh π is successively refined, the associated sequence of interpolants of a sufficiently smooth function f converges to f (Theorem 5). This

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scheme is the first *convergent* bicubic spline interpolation scheme applicable to a wide variety of rectangular polygons.

In Section 2 we discuss a univariate cubic spline interpolation scheme which is algebraically well defined for meshes containing an *even* number of mesh points and which is inconsistent for an *odd* number of mesh points (Theorem 2). In the former case it is also analytically well set (Theorem 3). This scheme is then used in the derivation of our main results given in Sections 3 and 4.

2. Univariate Cubic Splines of Interpolation

Let us first establish some notation. Let (I, π) denote a uniformly partitioned interval $I = [a, b]$ where $\pi : a = x_0 < \cdots < x_m = b$. Let $h = (b - a)/m$ be the mesh spacing. The $2(m + 1)$ dimensional *smooth Hermite space* of piecewise cubic polynomials in $C^1[I]$ is denoted by $H^2(I, \pi)$ [6]. For a given function f, the *smooth Hermite interpolant* u_f of f, is the unique element in $H^2(I, \pi)$ satisfying

$$\begin{aligned} &\text{A.} \quad u_f(x_i) = f(x_i), \qquad 0 \leqslant i \leqslant m, \\ &\text{B.} \quad u_f'(x_i) = f'(x_i), \qquad 0 \leqslant i \leqslant m. \end{aligned} \tag{1}$$

Thus the $2(m + 1)$ values $\{u_f(x_i), u_f'(x_i), 0 \leqslant i \leqslant m\}$ are the "coordinates" of u_f relative to the standard interpolating basis for $H^2(I, \pi)$ [6]. In fact, for $x_{i-1} \leqslant x \leqslant x_i$,

$$u_f(x) = H_1(\bar{x}) f_{i-1} + H_2(\bar{x}) f_i + H_3(\bar{x}) f'_{i-1} + H_4(\bar{x}) f_i' \tag{2}$$

where $\bar{x} = x - x_{i-1}$ and the cubics $H_j(\bar{x})$ are given, e.g., in [10].

Of primary concern here is the $(m + 3)$-dimensional subspace $S_1^2(I, \pi)$ of cubic splines s defined by

$$S_1^2(I, \pi) \equiv H^2(I, \pi) \cap C^2(I). \tag{3}$$

The more stringent continuity requirement, $s \in C^2(I)$, is equivalent [1, 3] to the following set of linear constraints on the "Hermite coordinates" of s:

$$h[s'_{i-1} + 4s_i' + s'_{i+1}] = 3[s_{i+1} - s_{i-1}], \qquad 1 \leqslant i \leqslant m - 1. \tag{4}$$

Since $\dim S_1^2(I, \pi) = m + 3$, specifying a subset of $(m + 3)$ Hermite coordinates which uniquely determine *all* Hermite coordinates (or equivalently satisfy (4)) is equivalent to defining an algebraically well set interpolation scheme. We consider three interpolation schemes or *splines of interpolation* denoted s_f, r_f, and v_f which are given in the following

DEFINITION 1. Given $\{f(x_i), f'(x_i) : 0 \leqslant i \leqslant m\}$ we define three splines of interpolation, denoted respectively by s_f, r_f, and v_f, as the unique elements in $S^2(I, \pi)$ satisfying the following

$$\begin{aligned} &\textit{Scheme A:} && s_f(x_i) = f(x_i), && 0 \leqslant i \leqslant m,\\ & && s_f'(x_j) = f'(x_j), && j = 0, m.\\ &\textit{Scheme B:} && r_f(x_j) = f(x_j), && j = 0, 1,\\ & && r_f'(x_i) = f'(x_i), && 0 \leqslant i \leqslant m.\\ &\textit{Scheme C:} && v_f(x_j) = f(x_j), && j = 0, m,\\ & && v_f'(x_i) = f'(x_i), && 0 \leqslant i \leqslant m \quad (m \text{ odd}). \end{aligned} \tag{5}$$

Remark. The cubic spline $s_f(x)$ in scheme A is the "usual" cubic spline interpolant of $f(x)$. (See, e.g., [1, 3, 5, 9]). The cubic spline $r_f(x)$ is a spline of *extrapolation* and is discussed in [7] and [8]. By contrast, $v_f(x)$ is well defined *only if m is odd* as we show in Theorem 2, below.

Our primary concern in this section is to derive error bounds for the three types of interpolating splines defined above, subject to errors in the given data (5). We first state results which are contained in [7, 8] and which yield (Corollaries 1 and 2) the desired error bounds for schemes A and B.

THEOREM 1 [7, 8]. *Let $f \in C^5[I]$ and, for given ξ_i and η_j, let*

(A) *$s(x)$ be the unique cubic spline (scheme A) satisfying*

$$s(x_i) = f(x_i) + \xi_i,\ 0 \leqslant i \leqslant m,\ s'(x_j) = f'(x_j) + \eta_j,\ j = 0, m;$$

(B) *$r(x)$ be the unique cubic spline (scheme B) satisfying*

$$r(x_i) = f(x_i) + \xi_i,\ i = 0, 1,\ r'(x_j) = f'(x_j) + \eta_j,\ 0 \leqslant j \leqslant m.$$

If $|\xi_i| \leqslant K_1$ and $|\eta_i| \leqslant K_2$, then

$$|(s - f)'(x_i)| \leqslant (1/60)\|f^{(5)}\| h^4 + (6/h) K_1 + K_2$$

and

$$|(r - f)(x_i)| \leqslant ((b - a)/180)\|f^{(5)}\| h^4 + 3K_1 + (b - a) K_2. \tag{6B}$$

From [10; Eqs. (9) and (11)] and (6A, B) above, we have

COROLLARY 1. $\|s - f\|_\infty = O(h^4 + K_1 + K_2 h)$.

COROLLARY 2. $\| r - f \|_\infty = O(h^4 + K_1 + K_2)$.

We next establish existence and uniqueness of scheme C.

THEOREM 2. *Let values* g_i, $i = 0, m$ *and* g_i', $0 \leqslant i \leqslant m$ *be given. There exists a unique spline* $v_g \in S_1^2(I, \pi)$ *such that*

$$v_g(x_i) = g_i, \; i = 0, m \qquad \text{and} \quad v_g'(x_i) = g'(x_i), \; 0 \leqslant i \leqslant m, \tag{7}$$

if and only if m is odd.

Proof. From (4) the Hermite coordinates of $v \equiv v_g$ (if it exists) must satisfy

$$A\mathbf{V} = \mathbf{K} \tag{8}$$

where

$$A = \begin{bmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{bmatrix}, \qquad \mathbf{V} = (v_1, ..., v_{m-1}) \quad \text{and}$$

$$\mathbf{K}_i = \begin{cases} h(g_0' + 4g_i' + g_2') + 3g_0, & i = 1, \\ h(g_{i-1}' + 4g_i' + g_{i+1}'), & 2 \leqslant i \leqslant m-2, \\ h(g_{m-2}' + 4g_{m-1}' + g_m') - 3g_m, & i = m-1. \end{cases} \tag{9}$$

If m is odd (even number of mesh points) then it is easily verified that A is nonsingular and in fact

$$A^{-1} = S - S^t \qquad \text{where} \quad S = \begin{bmatrix} 0 & & & & & & \\ 1 & \ddots & & & \bigcirc & & \\ 0 & 0 & \ddots & & & & \\ 1 & 0 & 1 & \ddots & & & \\ 0 & 0 & 0 & 0 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{10}$$

However, if m is even (odd number of mesh points) then A is in fact an odd ordered skew-symmetric matrix and hence, singular. Furthermore, for m even, the sum of the odd number rows in A is zero which implies the system is *inconsistent* unless $\sum_{j=1}^{m/2} \mathbf{K}_{2j-1} = 0$. Obviously then there exist values g_i, $i = 0, m$ and g_i', $0 \leqslant i \leqslant m$ such that (10) is *inconsistent* and the proof of Theorems 2 is complete.

Analogous to Theorem 1 we have

THEOREM 3. *For given* ξ_i, η_i, *and* $f \in C^5[I]$, *let v be the unique spline*

(Scheme C) in $S^2(I, \pi)$, *(π with an even number of mesh points) such that* $v_i = f_i + \xi_i$, $i = 0, m$ *and* $v_i' = f_i' + \eta_i$, $0 \leqslant i \leqslant m$. If $|\xi_i| \leqslant K_1$ *and* $|\eta_i| \leqslant K_2$, *then*

$$|v_i - f_i| \leqslant \frac{(b-a)}{60} \|f^{(5)}\| h^4 + 3[K_1 + (b-a) K_2] \tag{11}$$

for $1 \leqslant i \leqslant m - 1$.

Proof. Analogous to [10; Eqs. (6) and (7)] it follows from (8) that for $K_1 = K_2 = 0$ we have

$$A\mathbf{E} = \mathbf{Z} \tag{12}$$

where $\mathbf{E}_i = v_{i+1} - f_{i+1}$ and $\mathbf{Z}_i = 1/30 f^{(5)}(\theta_i)\, h^5$, $x_i \leqslant \theta_i \leqslant x_{i+1}$. From (10) it follows that

$$\|\mathbf{E}\|_\infty \leqslant (m-1)/2 \, \|\mathbf{Z}\|_\infty \leqslant (b-a)/60 \, \|f^{(5)}\| \, h^4. \tag{13}$$

Perturbations due to ξ_i and η_i modify (12) as

$$A\hat{\mathbf{E}} = \mathbf{Z} + \boldsymbol{\phi} + \boldsymbol{\psi} \tag{14}$$

where from (9) $\psi_i = h(\eta_{i-1} + 4\eta_i + \eta_{i+1})$, $1 \leqslant i \leqslant m-1$ and $\phi_i = 3\xi_0$ for $i = 1$; $= 0$ for $2 \leqslant i \leqslant m-2$; and $= 3\xi_m$ for $i = m - 1$. From (10) it follows that

$$\|\hat{\mathbf{E}}\|_\infty \leqslant \|\mathbf{E}\|_\infty + \|A^{-1}\boldsymbol{\phi}\| + \|A^{-1}\| \cdot \|\boldsymbol{\psi}\| \leqslant \|\mathbf{E}\|_\infty + 3K_1 + ((m-1)/2)6hK_2,$$

from which (11) follows.

COROLLARY 3. $\|v - f\|_\infty = O(h^4 + K_1 + K_2)$.

Proof. As in [10] write

$$v - f = (v - u_f) + (u_f - f). \tag{15}$$

The second term is $O(h^4)$ from [6]. The first term

$$(v - u_f)(x) = \hat{\mathbf{E}}_i H_1(\bar{x}) + \hat{\mathbf{E}}_{i+1} H_2(\bar{x}) + \eta_i H_3(\bar{x}) + \eta_{i+1} H_4(\bar{x}) \tag{16}$$

for $x \in [x_i, x_{i+1}]$. Now as $h \to 0$ $H_i(\bar{x}) = O(1)$ for $i = 1, 2$ and $= O(h)$ for $i = 3, 4$. Hence from (11), $\|(v - u_f)\| = O(h^4 + K_1 + K_2)$, and the proof of Corollary 3 is complete.

3. Bicubic Spline Interpolation Scheme

In [7; Theorem 4] it was first establish that if $(\mathscr{R}, \pi)$ is a partitioned *rectangular polygon* then

$$\dim S_1{}^2(\mathscr{R}, \pi) = M + B + 8 \equiv d,$$

where M is the number of mesh points and B is the number of boundary mesh points. This compares with [6]

$$\dim H^2(\mathscr{R}, \pi) = 4M$$

for the smooth Hermite space of order 2. Since $S_1{}^2(\mathscr{R}, \pi)$ is a subspace of $H^2(\mathscr{R}, \pi)$, one way of specifying an algebraically well set interpolation scheme is to choose (if possible) a subset of d Hermite coordinates which span $S_1{}^2(\mathscr{R}, \pi)$. We now state and prove our main result of this section.

THEOREM 4. *Let $(\mathscr{R}, \pi)$ be a uniformly partitioned rectangular polygon such that each mesh line of π contains an even number of mesh points. Let the following values be given.*[1]

(i) $f_{ij}^{(1,1)}$ *at each mesh point (x_i, y_j),*

(ii) $f_{ij}^{(1,0)}$ *at the end points of each* vertical *mesh line,*

(iii) $f_{ij}^{(0,1)}$ *at the end points of each* horizontal *mesh line,*

(iv) f_{ij} *at the four corners of an amenable* [7] *set of corner points.*

Then there exists a unique spline $v_f \in S_1{}^2(\mathscr{R}, \pi)$ such that

$$v_f^{(r,s)}(x_i, y_j) = f_{ij}^{(r,s)} \tag{17}$$

at the respective values and points specified in (i)–(iv).

Remark. Note that *not* all rectangular polygons can be partitioned such as to satisfy the above hypothesis. For example in Fig. 1, the mesh lines $\overline{AB}$ and $\overline{CD}$ must each contain an even number of mesh points, but this implies the mesh line $\overline{EF}$ contains an *odd* number of mesh points. Hence the region in Fig. 1 does *not* satisfy the hypothesis, irrespective of π.

Proof. Basically, the task is to show that the smooth Hermite coordinates $\{v_f^{(r,s)} : 0 \leqslant r, s \leqslant 1\}$ for each point (x_i, y_j) in π can be derived in a unique manner. This is indeed possible with the use of the three univariate splines of interpolation delineated in Section 2. The general proof is analogous to the

[1] $f_{ij}^{(k,l)} \equiv (\partial f^{k+l}/\partial x^k \, \partial y^l)(x_i, y_j)$.

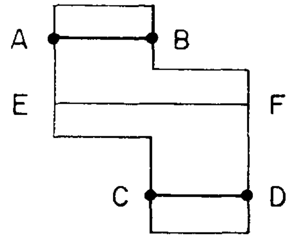

FIGURE 1

proof of [7; Theorem 3] and will be omitted in lieu of the following proof for a specific $\mathscr{R}$.

Consider the region in Fig. 2. Of primary importance is the fact that v_f (if it exists) belongs to $C^{(2,2)}[\mathscr{R}]$[7, Theorem 2] and hence $v_f(x_i, y)$, $v_f(x, y_j)$, $v_f^{(1,0)}(x_i, y)$ and $v_f^{(0,1)}(x, y_j)$ are all univariate cubic splines for each i, j.

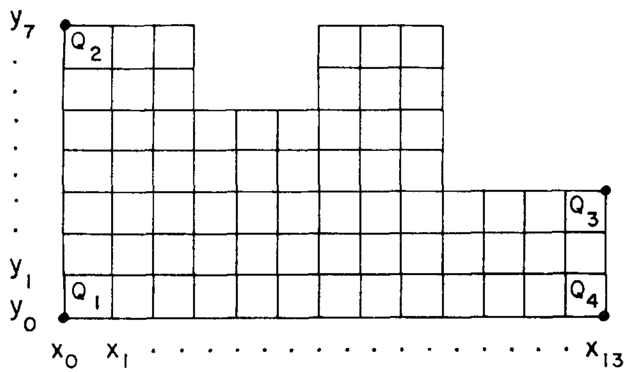

FIGURE 2

Step 1. Using the notation of Section 2 we construct a scheme-C spline of interpolation along each horizontal and vertical mesh line to compute $v_{fij}^{(1,0)}$ anx $v_{fij}^{(0,1)}$ at each mesh point. For example

$$v_{f01}^{(0,1)},\ v_{f13,1}^{(0,1)} \text{ and } v_{fi1}^{(1,1)}, \qquad 0 \leqslant i \leqslant 13 \tag{18}$$

are known, and hence by Theorem 2 the values $v_{fi1}^{(0,1)}$, $0 < i < 13$ are uniquely determined.

Step 2. Construct a scheme-C spline along $x = x_0$, $x = x_{13}$, and $y = y_0$ to compute v_{fij} along these lines. For example

$$v_{f00},\ v_{f07} \text{ and } v_{f0j}^{(0,1)}, \qquad 0 \leqslant j \leqslant 7, \tag{19}$$

are known and hence by Theorem 2 the values v_{f0j}, $0 < j < 7$, are uniquely determined.

Step 3. Construct a scheme-C spline along $y = y_j$, $j = 1, 2, 3$, to compute v_{fij} along these lines. For example,

$$v_{f02}, \; v_{f13,2} \text{ and } v_{fi,2}^{(1,0)}, \qquad 0 \leqslant i \leqslant 13, \tag{20}$$

are known and hence the values v_{fi2}, $0 < i < 13$ are uniquely determined.

Step 4. Construct a scheme-B spline along $x = x_i$ for $y \geqslant y_2$ and $1 \leqslant i \leqslant 9$ to compute v_{fij} along these lines. For example,

$$v_{f72}, \; v_{f73}, \; v_{f7j}^{(0,1)}, \qquad 2 \leqslant j \leqslant 6, \tag{21}$$

are known and hence from [8] the values v_{f7j}, $4 \leqslant j \leqslant 6$ are uniquely determined.

Hence the Hermite coordinates are uniquely determined at each $(x_i, y_j) \in \pi$ such that $v_f \in C^{(2,2)}[\mathscr{R}]$. Note that the consistency of these values follows as in [7].

4. Convergence Theorem

The following theorem establishes that the interpolation scheme in Section 3 is *analytically well set* for a wide class of sequences of partitionings. Moreover the convergence is fourth order and hence comparable to de Boor's bicubic spline interpolant for rectangular regions.

THEOREM 5. *Let $(\mathscr{R}, \pi)$ be a uniformly partitioned rectangular polygon. Assume that each mesh line of π contains an even number of mesh points (see the remark after Theorem 4). Let v_f be the spline interpolant of $f \in C^{(5,1)}[\mathscr{R}] \cap C^{(1,5)}[\mathscr{R}]$ described in Theorem 4. Then as $h \to 0$*

$$\|(v_f - f)^{(k+l)}\|_\infty = O(h^{4-(k+l)}), \qquad 0 \leqslant k + l \leqslant 3. \tag{22}$$

Proof. Consider the error function $e(x, y) = v_f(x, y) - f(x, y)$. As in [8] we write

$$e(x, y) = \{v_f(x, y) - u_f(x, y)\} + \{u_f(x, y) - f(x, y)\} \tag{23}$$

where $u_f(x, y) \in H^2(\mathscr{R}, \pi)$ is the *smooth Hermite interpolant* to f, [6]. From [6; p. 249] we have

$$\|(u_f - f)^{(k,l)}\|_\infty = O(h^{4-(k+l)}), \qquad 0 \leqslant k + l \leqslant 3, \quad \text{as} \quad h \to 0. \tag{24}$$

Next we note that for $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$,

$$|(v_f - u_f)^{(k,l)}(x, y)| \leqslant \sum_{n=1}^{2} \sum_{m=1}^{2} \{| e_{mn} | \cdot | H_n^{(k)}(\bar{x}) G_m^{(l)}(\bar{y})| + | e_{mn}^{(1,0)} | \cdot | H_{n+2}^{(k)}(\bar{x}) G_m^{(l)}(\bar{y})| + | e_{mn}^{(0,1)} | \cdot | H_n^{(k)}(\bar{x}) G_{m+2}^{(l)}(\bar{y})|\} \tag{25}$$

where $e_{mn}^{(k,l)} \equiv e_{i+m-1,j+n-1}^{(k,l)}$, $\bar{x} = x - x_i$, $\bar{y} = y - y_j$ and the $H_n(\bar{x})$ and $G_m(\bar{y})$ are given in [10]. Note that $G_m(y) \equiv H_m(y)$ for a uniform mesh and $H_m^{(k)}(x) = O(h^{-k})$ for $m = 1, 2$ and $H_m^{(k)}(x) = O(h^{1-k})$ for $m = 3, 4$. Hence to bound (25) as $O(h^{4-(k+l)})$ we need only show that

$$e_{mn}^{(k,l)} = O(h^{4-(k+l)}), \qquad 0 \leqslant k + l \leqslant 1. \tag{26}$$

We verify (26) for the region $\mathscr{R}$ in Fig. 1. The general case follows in an analogous manner.

Consider the four step construction in the proof of Theorem 4. Let us determine the propagation of errors.

Steps 1–2. In step 1 and step 2 all the values were known exactly and hence from Corollary 3 with $K_1 = K_2 = 0$ we have that $e_{mn}^{(1,0)}$ and $e_{mn}^{(0,1)}$ are $O(h^4)$ as $h \to 0$ at *all* mesh points and e_{mn} is $O(h^4)$ at those points considered in step 2.

Step 3. Applying Corollary 3 with K_1 and K_2 as $O(h^4)$ we note that e_{mn} is $O(h^4)$ at those points considered.

Step 4. Applying Corollary 2 with K_1 and K_2 as $O(h^4)$ we note that e_{mn} is $O(h^4)$ at those points considered.

Hence (26) is established and the proof is complete.

5. Alternate Bicubic Schemes

The coordinates or parameters chosen in Theorem 4 are by no means the only possible coordinates for an algebraically well set scheme. The reason for this choice was the applicability to a broad class of *rectangular polygons*. It's also a more "symmetric" choice of parameters than those in Examples 1 or 2 below. Given a *specific* region it is not difficult in light of Theorems 1 and 3 and their corollaries to specify other algebraically and analytically well set interpolation schemes. However, invariably such a scheme is quite dependent on the specific region. We now delineate (without proof) several such schemes.

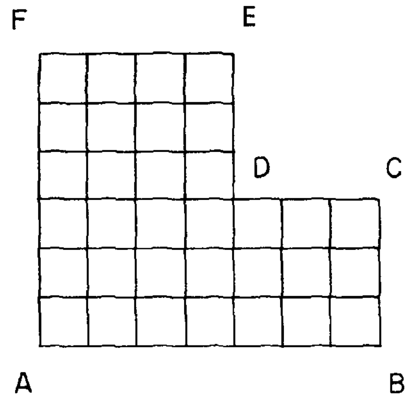

FIGURE 3

EXAMPLE 1 [5, 8]. Let (L, π) be a partitioned L-shaped region (Fig. 3). There exists a unique bicubic spline s which interpolates to $f \in C^p[L]$ at each mesh point, to the normal derivative at each boundary mesh point not on $\overline{CD}$, and to $f^{(1,1)}$ at the corners A, B, F, and E as well as along CD excluding the point D. If $p = 4$ and the mesh ratio[2] is bounded then

$$\|(s-f)^{(k,l)}\| = O(h^{3-(k+l)}), \qquad 0 \leqslant k+l \leqslant 3$$

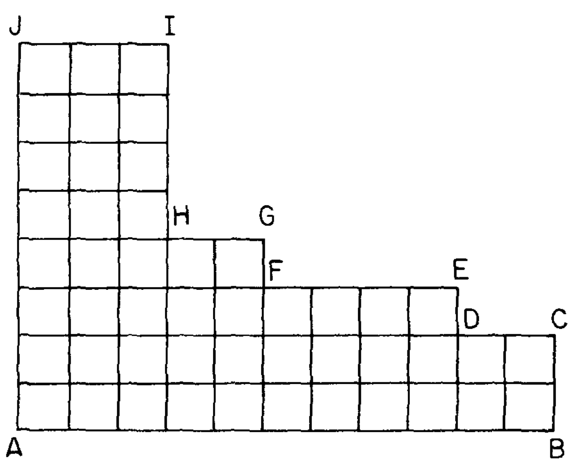

FIGURE 4

[2] If π is a *nonuniform* mesh then the mesh ratio is by definition

$$\beta \equiv \max_{ij}\{x_i - x_{i-1}, y_j - y_{j-1}\}/\min\{x_i - x_{i-1}, y_j - y_{j-1}\}.$$

Note that the numerator is the maximum mesh spacing h and $\beta = 1$ for π uniform.

and if $p = 5$ and π is uniform then

$$\|(s-f)^{(k,l)}\| = O(h^{4-(k+l)}), \qquad 0 \leqslant k + l \leqslant 3.$$

The proof of the above result can easily be modified to yield the following

EXAMPLE 2. Let (T, π) be a partitioned "step" region (Fig. 4). There exists a unique bicubic spline s which interpolates to $f \in C^5[T]$ at each mesh point; to the normal derivative at each boundary mesh point *not* on $\overline{CD}$, $\overline{EF}$, or $\overline{GH}$; and to $f^{(1,1)}$ at the corners A, B, J, I as well as along $\overline{CD}$, $\overline{EF}$, $\overline{GH}$ excluding the points D, F, and H. If the mesh ratio is bounded then

$$\|(s-f)^{(k,l)}\| = O(h^{3-(k+l)}), \qquad 0 \leqslant k + l \leqslant 3. \tag{27}$$

If π is uniform then

$$\|(s-f)^{(k,l)}\| = O(h^{4-(k+l)}), \qquad 0 \leqslant k + l \leqslant 3. \tag{28}$$

Our final example indicates how the main interpolation scheme (Theorem 4) can often be modified to handle regions such as given in Fig. 1.

EXAMPLE 3. Let $(\mathscr{R}, \pi)$ be the region in Fig. 5 partitioned by $\pi = \pi_1 \times \pi_2$ where $\pi_1 : x_0 < x_1 < \cdots < x_{2p-1} < \cdots < x_{2n}$, $\pi_2 : y_0 < y_1 < \cdots < y_{2m-1}$ are uniform partitions and each vertical mesh line contains an even number of mesh points. (See the remark after Theorem 4.)

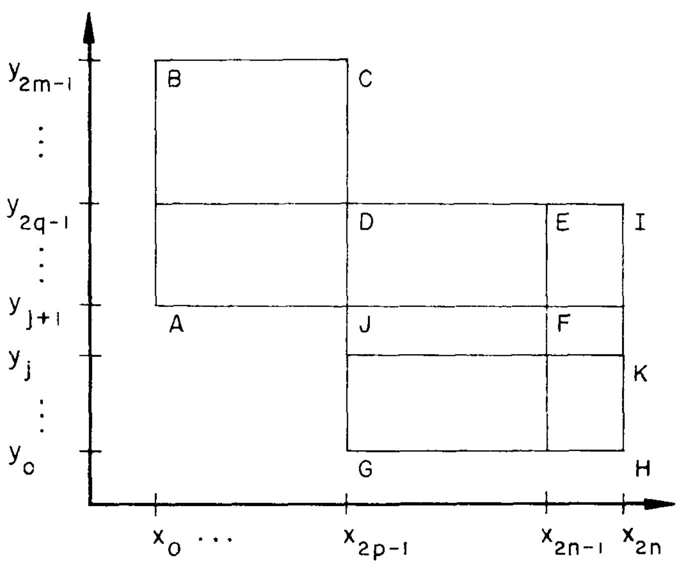

FIGURE 5

There exists a unique bicubic spline s which interpolates to $f \in C^5[\mathscr{R}]$ at A, B, C, and G; to the tangential derivative $f^{(1,0)}$ along $\overline{BC}$, $\overline{GH}$, $\overline{DI}$, and $\overline{AJ}$ excluding the reentrant corners D and J; to $f^{(0,1)}$ along $\overline{AB}$, $\overline{CD}$, $\overline{JG}$, $\overline{EF}$, and $\overline{KH}$ excluding the reentrant corners D and J: and $f_{ij}^{(1,1)}$ at each mesh point of π. Further,

$$\|(s-f)^{(k,l)}\| = O(h^{4-(k+l)}), \qquad 0 \leqslant k + l \leqslant 3. \tag{29}$$

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